Risk-Averse Optimal Control

Sassan Mokhtar

May 15, 2020

RWTH AACHEN UNIVERSITY

Outline

Table of Contents

Merton's Portfolio problem

The value of the portfolio $X(t)$, consists of a risky assets, $S(t) = u(t)X(t)$, and risk less bonds with risk-free rate r, $B(t) = (1 - u(t))X(t)$, where $u(t) \in [0, 1]$ is the control and

$$dS = a_1 S dt + a_2 S dW, \quad dB = rBdt$$

According to the self-financing considerations,

$$\begin{aligned} dX_s &= (a_1 u_s + r(1 - u_s))X_s ds + a_2 u_s X_s dW_s, \quad \text{for } s > t \\ X_t &= x \end{aligned}$$

For a given function g the cost function is defined as,

$$C_{t,x}(u) = \mathbb{E}[g(X(T))|X_t = x]$$

The goal is to determine the Markov control function that maximizes the cost function.

Merton's Portfolio problem

The value function is: $V(t,x) = \max_u C_{t,x}(u)$, which solves the HJB equation:

$$V_t + \mathcal{H}(t,x,V_x,V_{xx}) = 0$$
$$V(T,x) = g(x)$$

for the Hamiltonian:

$$\mathcal{H}(t,x,p,w) = \max_u (a_1 u + r(1-u)xp + \frac{{a_2}^2 u^2}{2} x^2 w)$$

→ Solving this equation gives us the optimal strategy, which maximizes the expected return of the risk neutral investor.

Problem setting

What about the risk-averse investor?

- Goal: To extend dynamic optimal control problems, which are risk-neutral, to a risk-averse environment.
  $\rightarrow$ In this setting, a decision maker suffers an uncertain amount of cost and his goal is to manage and minimize the total costs.
- Risk-averse decision maker: A person who prefer a certain outcome with a lower pay-off over an uncertain outcome with a higher pay-off.
- Risk measures: Therefore, we need to introduce some risk measures to quantify the preferences of the decision makers.

Table of Contents

Value at Risk (VaR)

- A risk measure which has been widely used since 1980. In the late 1980s, it was integrated by JP Morgan on a firm-wide level into its risk-management system.
- VaR is defined as the minimum level of loss at a given confidence level for a predefined time horizon.
  $\rightarrow$ A portfolio with a 1-day 99% VaR equal to 1 million means over the horizon of 1 day, the portfolio may lose more than 1 million with probability equal to 1%.
- The VaR at level $\hat{\beta} \in (0,1)$ of the random variable X is defined as:

$$\text{VaR}_{\hat{\beta}}(X) = \sup_x \{x | P(X \leq x) \leq \hat{\beta}\}$$

Value at Risk (VaR)

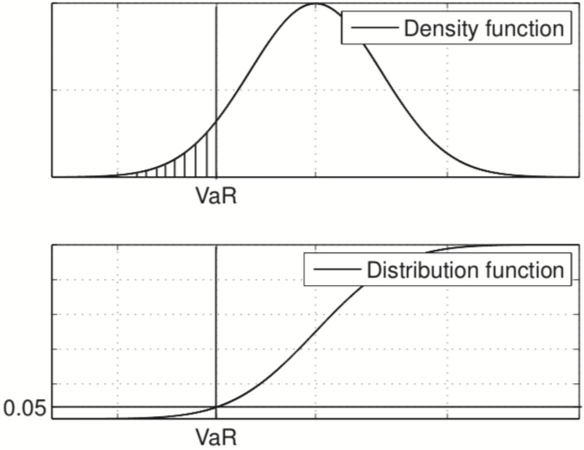

Figure: The VaR at 95% risk level of the random variable X

Coherent Risk Measures

Can we find a set of desirable properties that a risk measure should satisfy?
$\rightarrow$ An answer is given by Artzner et. al. [1]. They provide an axiomatic definition of a functional which they call a coherent risk measure $(\mathcal{R})$.

1. *Monotonicity:* if $Y \leq Y'$, then $\mathcal{R}(Y) \leq \mathcal{R}(Y')$
2. *Subadditivity:* $\mathcal{R}(Y + Y') \leq \mathcal{R}(Y) + \mathcal{R}(Y')$
3. *Translation equivariance:* $\mathcal{R}(Y + c) = \mathcal{R}(Y) + c$ for $c \in \mathbb{R}$
4. *Positive homogeneity:* $\mathcal{R}(\lambda Y) = \lambda.\mathcal{R}(Y)$ for $\lambda > 0$

VaR does not satisfies Subadditivity property.
$\rightarrow$ Next, we introduce a risk measure, which satisfies the above axioms and thus is a coherent risk measure.

Entropic Value at Risk (EVaR) Ahmadi-Javid [5]

- Entropic VaR: Coherent risk measure that is the tightest possible upper bound obtained from the Chernoff inequality.
- Chernoff inequality: For any constant γ and the random variable X, whose moment generating function $M_X(\ell)$ exists for $\ell > 0$; it holds,

$$P(X \geq \gamma) \leq e^{-\ell\gamma} M_X(\ell) = e^{-\ell\gamma} \mathbb{E}[e^{\ell X}]$$

- By solving the equation $e^{-\ell\gamma} M_X(\ell) = \hat{\beta}$ with respect to γ for $\hat{\beta} \in [0, 1]$,

$$\gamma(\hat{\beta}, \ell) = \frac{1}{\ell} log\left(\frac{M_X(\ell)}{\hat{\beta}}\right)$$

  for which we have $P(X \geq \gamma(\hat{\beta})) \leq \hat{\beta}$. In fact, for each $\ell > 0$, $\gamma(\hat{\beta})$ is an upper bound for $\mathrm{VaR}_{1-\hat{\beta}}$. We define the best upper bound of this type as EVaR.
- By changing variable $\beta = -log\hat{\beta}$, we obtain the following formula for EVaR, for $\beta \in [0, \infty)$

$$\mathrm{EVaR}_\beta(X) = \inf_{\ell > 0} \frac{1}{\ell}\left(\beta + \log \mathbb{E}[e^{\ell X}]\right)$$

EVaR of Gaussians

For a normally distributed random variable $X \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma \geq 0$ and $a, b \in \mathbb{R}$ it holds that:

$$\text{EVaR}_\beta(a + bX) = a + b\mu + \sigma|b|\sqrt{2\beta} \tag{1}$$

Proof:

It holds that $\mathbb{E}[e^{\ell X}] = \exp(\mu\ell + \frac{1}{2}\ell^2\sigma^2)$ and thus,

$$\frac{1}{\ell}\beta + \frac{1}{\ell}\log\left(e^{\mu\ell + \frac{1}{2}\ell^2\sigma^2}\right) = \frac{1}{\ell}\beta + \mu + \frac{1}{2}\ell\sigma^2$$

which attains its minimum at $\ell^* = \frac{1}{\sigma}\sqrt{2\beta}$. Thus,

$$\text{EVaR}_\beta(X) = \frac{1}{\ell^*}\beta + \mu + \frac{1}{2}\ell^*\sigma^2 = \mu + \sigma\sqrt{2\beta}$$

Finally, using the fact that $a + bX \sim \mathcal{N}(a + b\mu, b^2\sigma^2)$ terminates the proof.

Nested EVaR

- Classical, non-nested, risk measures only asses the accumulated position at time T, so that the risk manager can never intervene.
- Thus, we introduce nested risk measures which allow quantifying risk via dynamic programming equations.

Definition 1 (Nested risk measures)

For the random variable $Y = (Y_t)_{t\in[0,T]}$ adapted to the filtration $\mathcal{F}_t$, the risk measure $\mathcal{R}$, and the partition $\mathcal{P} = [0 = t_0 < ... < t_i < ... < t_n = T]$, the nested risk measures is defined as:

$$\text{if } i = n-1: \quad \mathcal{R}^{\{t_j\}_{j=n-1}^n}(Y|\mathcal{F}_{t_{n-1}}) := \mathcal{R}^{t_{n-1}}(Y|\mathcal{F}_{t_{n-1}})$$

$$\text{if } i < n-1: \quad \mathcal{R}^{\{t_j\}_{j=i}^n}(Y|\mathcal{F}_{t_i}) := \mathcal{R}^{t_i}\left(\mathcal{R}^{\{t_j\}_{j=i+1}^n}(Y|\mathcal{F}_{t_{i+1}})\middle|\mathcal{F}_{t_i}\right)$$

Nested Risk Measures

Definition 2 (Nested Entropic Value at Risk)

For the partition $\mathcal{P}$ and a vector of risk levels $\beta := (\beta_{t_i}\Delta t_i, ..., \beta_{t_{n-1}}\Delta t_{n-1})$, the nested EVaR is:

$$\text{if } i = n-1:$$

$$nEVaR_{\beta}^{\{t_j\}_{j=n-1}^{n}}(Y|\mathcal{F}_{t_{n-1}}) := EVaR_{\beta_{t_{n-1}}.\Delta t_{n-1}}(Y|\mathcal{F}_{t_{n-1}})$$

$$\text{if } i < n-1:$$

$$nEVaR_{\beta}^{\{t_j\}_{j=i}^{n}}(Y|\mathcal{F}_{t_i}) := EVaR_{\beta_{t_i}.\Delta t_i}\left(nEVaR_{\beta}^{\{t_j\}_{j=i+1}^{n}}(Y|\mathcal{F}_{t_{i+1}})\middle|\mathcal{F}_{t_i}\right)$$

Often, the risk evaluation of $\mathcal{R}^{0:T}(Y_T)$ of the terminal value Y_T of some stochastic process Y is of interest.

Nested Risk Measures

Proposition 1 (Pichler and Schlotter [2])

For $Y_T = Y_{t_i} + \sum_{j=i}^{n-1} \Delta Y_{t_j}$ *and the partition* $\mathcal{P} = [0 = t_0 < \ldots < t_i < \ldots < t_n = T]$, *it holds that*

$$\begin{aligned} &\textit{if } i = n-1: \\ &\qquad \mathcal{R}^{\{t_j\}_{j=n-1}^{n}}(Y_T|\mathcal{F}_{t_{n-1}}) := Y_{t_{n-1}} + \mathcal{R}^{t_{n-1}}(\Delta Y_{t_{n-1}}|\mathcal{F}_{t_{n-1}}) \\ &\textit{if } i < n-1: \\ &\qquad \mathcal{R}^{\{t_j\}_{j=i}^{n}}(Y_T|\mathcal{F}_{t_i}) := \mathcal{R}^{t_i}\left(\mathcal{R}^{\{t_j\}_{j=i+1}^{n}}(Y_T|\mathcal{F}_{t_{i+1}})\middle|\mathcal{F}_{t_i}\right) \end{aligned} \tag{2}$$

Therefore, it is sufficient to study conditional risk evaluations of increments.

Nested EVaR for Gaussian Random Walk

Let $W = (W_t)_{t\in\mathcal{P}}$ be a Wiener process evaluated on the partition $\mathcal{P}$. Also, let $\beta := (\beta_{t_0}\Delta t_0, ..., \beta_{t_{n-1}}\Delta t_{n-1})$ be a vector of risk levels. Then,

$$\text{nEVaR}_\beta^{\mathcal{P}}(W_T) = \sum_{i=0}^{n-1} \Delta t_i \sqrt{2\beta_{t_i}}$$

Proof:

Note that $W_{t_{i+1}} - W_{t_i} \sim \mathcal{N}(0, t_{i+1} - t_i)$ and by (1), the conditional EVaR is:

$$\text{EVaR}_{\beta_{t_i}\Delta t_i}(W_{t_{i+1}}|W_{t_i}) = W_{t_i} + \sqrt{\Delta t_i}\sqrt{2\beta_{t_i}\Delta t_i}$$

Iterating as in (2), shows

$$\text{nEVaR}_\beta^{\mathcal{P}}(W_T) = \sum_{i=0}^{n-1} \sqrt{\Delta t_i}\sqrt{2\beta_{t_i}\Delta t_i} = \sum_{i=0}^{n-1} \Delta t_i \sqrt{2\beta_{t_i}}$$

Nested EVaR in Continuous Time

- Up to now, the nested risk measures in discrete time on partition $\mathcal{P} = [0 = t_0 < \ldots < t_n = T]$ of the interval $\mathcal{T} = [0, T]$ is considered.
- The next step is to refine the partition to obtain a limit of the nEVaR in continuous time.

$\rightarrow$ First, extend the vector of risk levels to continuous time by introducing the function $\beta : [0, T] \rightarrow [0, \infty)$ called the risk rate. Then, extend the definition of the nEVaR to risk rates $\beta(.)$.

Definition 3

For a given finite partition $\mathcal{P}$ and the Riemann integrable risk rate $\beta(.)$, the nEVaR is defined as,

$$nEVaR^{\mathcal{P}}_{\beta(.)}(Y) := nEVaR^{\mathcal{P}}_{\hat{\beta}}(Y)$$

where the vector of risk levels is $\hat{\beta} := (\beta(t_0)\Delta t_0, \ldots, \beta(t_{n-1})\Delta t_{n-1})$.

Nested EVaR in Continuous Time

Given the Riemann integrable risk rate $\beta(.)$, it is time to investigate the relationship of nEVaR for different partitions.

Proposition 2 (Pichler and Schlotter [2])

Let Y be a random variable and $\beta : [0, T] \to [0, \infty)$ be a piecewise constant risk rate. For a partition $\mathcal{P}_1$, fine enough to contain all points of discontinuity of β and for every refinement $\mathcal{P}$ included in $\mathcal{P}_1$, the following inequality holds true:

$$nEVaR^{\mathcal{P}}_{\beta(.)}(Y) \leq nEVaR^{\mathcal{P}_1}_{\beta(.)}(Y)$$

Proof:

It is enough to consider the partitions $\mathcal{P}_1 = \{0 = t_0 < t_2 = T\}$ and $\mathcal{P} = \{0 = t_0 < t_1 < t_2 = T\}$ of $[0, T]$ and the constant risk rate $\beta(.)$. Then, the general case follows by induction.

Proof

According to the definition of the EVaR,

$$\mathrm{EVaR}_{\beta(t_1)\Delta t_1}(Y|\mathcal{F}_{t_1}) = \inf_{\ell} \frac{1}{\ell}\Big(\beta(t_1)\Delta t_1 + \log(\mathbb{E}[e^{\ell Y}\big|\mathcal{F}_{t_1}])\Big)$$

By nesting,

$$\mathrm{nEVaR}^{\mathcal{P}}_{\beta(.)}(Y) =$$

$$\inf_{x} \frac{1}{x}\Bigg(\beta(t_0)\Delta t_0 + \log \mathbb{E}\Bigg[\exp\Bigg(x\Big(\inf_{\ell} \frac{1}{\ell}(\beta(t_1)\Delta t_1 + \log(\mathbb{E}[e^{\ell Y}|\mathcal{F}_{t_1}]))\Big)\Bigg)\Bigg|\mathcal{F}_{t_0}\Bigg]\Bigg)$$

Choose $\ell = x$,

$$\begin{aligned}\mathrm{nEVaR}^{\mathcal{P}}_{\beta(.)} &\leq \inf_{x} \frac{1}{x}\Big(\beta(t_0)\Delta t_0 + \beta(t_1)\Delta t_1 + \log(\mathbb{E}[\mathbb{E}[e^{xY}|\mathcal{F}_{t_1}]|\mathcal{F}_{t_0}])\Big)\\ &= \mathrm{EVaR}_{\beta(t_0)\Delta t_0 + \beta(t_1)\Delta t_1}(Y)\end{aligned}$$

But $\beta(.)$ is constant. Thus,

$$\mathrm{nEVaR}^{\mathcal{P}}_{\beta(.)}(Y) \leq \mathrm{EVaR}_{\beta(t_0)(\Delta t_0 + \Delta t_1)}(Y) = \mathrm{nEVaR}^{\mathcal{P}_1}_{\beta(.)}(Y)$$

Nested EVaR in Continuous Time

Now, we are able to extend the nEVaR to continuous time:

Definition 4 (nEVaR in continuous time)

For $\beta(.)$ Riemann integrable,

$$nEVaR^{t:T}_{\beta(.)}(Y|\mathcal{F}_t) := essinf_{\mathcal{P},\tilde{\beta}(.)\geq\beta(.)} nEVaR^{\mathcal{P}}_{\tilde{\beta}(.)}(Y|\mathcal{F}_t)$$

where the infimum is among all partitions $\mathcal{P} \subset [t, T]$ and piecewise constant functions $\tilde{\beta}(.) \geq \beta(.)$

According to proposition (2), the essential infimum in the definition of the continuous nEVaR can be replaced by the limit of nonincreasing sequence.

Nested EVaR of Wiener Process

The nested EVaR of the Wiener process W on $\mathcal{T} = [0, T]$ for a risk rate $\beta : \mathcal{T} \to [0, \infty)$ is

$$\text{nEVaR}^{0:T}_{\beta(.)}(W_T) = \int_0^T \sqrt{2\beta(t)}dt$$

Proof:

Consider the partition $\mathcal{P}$ of size n of $\mathcal{T}$. By nesting:

$$\text{nEVaR}^{0:T}_{\beta(.)}(W_T) = \sum_{i=0}^{n-1} \sqrt{\Delta t_i}\sqrt{2\beta(t_i)\Delta t_i} = \sum_{i=0}^{n-1} \Delta t_i \sqrt{2\beta(t_i)}$$

Taking the limit $n \to \infty$ shows that

$$\text{nEVaR}^{0:T}_{\beta(.)}(W_T) = \int_0^T \sqrt{2\beta(t)}dt$$

Table of Contents

Infinitesimal Generator

- The infinitesimal generator is the differential operator describing the evolution of the system.
- Goal: To introduce the generator in the presence of risk $\rightarrow$ which extends the notion of the infinitesimal generator of Markov processes by replacing the expectation by a risk measure.
- This enables us to formulate and solve risk-averse control problems.

Infinitesimal generator

Recall the generator,

$$\mathcal{G}\phi(t,x) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E}\left[\phi(t+\Delta t, X_{t+\Delta t}) - \phi(t,x) \middle| X_t = x\right]$$

Then, for the SDE $dX_t = b(X_t,t)dt + \sigma(X_t,t)dW_t$ it holds:

$$\mathcal{G} = \frac{\partial}{\partial t} + b\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2}$$

Entropic Generator

Proposition 3 (Entropic generator, Pichler and Schlotter [2])

For the entropic generator:

$$\mathcal{G}\phi(t,x) := \lim_{\Delta t \to 0} \frac{1}{\Delta t} EvaR_{\beta.\Delta t}\left[\phi(t+\Delta t, X_{t+\Delta t}) - \phi(t,x) \middle| X_t = x\right]$$

It holds that

$$\mathcal{G}\phi = \frac{\partial}{\partial t}\phi + b\frac{\partial}{\partial x}\phi + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\phi + \sqrt{2\beta}\left|\sigma.\frac{\partial}{\partial x}\phi\right|$$

- The entropic risk generator can be decomposed as the sum of the classical generator and the nonlinear term $\sqrt{2\beta}\left|\sigma.\frac{\partial}{\partial x}\phi\right|$.
- This additional risk term can be interpreted as a directed drift term where the uncertain drift $\frac{\partial \phi}{\partial x}(t, X_t)$ is scaled with volatility σ.
- The coefficient $\sqrt{2\beta(.)}$ expresses risk aversion.

Sketch of Proof

Assumption 1

→ *Hölder continuity: Assume that there exist a* $\hat{C} > 0$ *and* $\eta \in (0, 1/2)$ *such that*

$$|\sigma(u, y) - \sigma(s, x)| \leq \hat{C}|u - s|^{\eta}$$

uniformly for all $x, y \in \mathbb{R}$.
→ *Also, assume that* $\phi \in C^{1,2}(\mathcal{T}, \mathbb{R})$ *such that* $\frac{\partial \phi}{\partial x}$ *is bounded.*

For the risk rate β, the risk generator based on the EVaR is:

$$\mathcal{G}\phi(t, x) := \lim_{h \to 0} \frac{1}{h}\left(\text{EVaR}_{\beta(t).h}\phi(t + h, X_{t+h}|X_t = x) - \phi(t, x)\right)$$

for those functions that the limit exists.

Sketch of Proof

Apply Itô formula:

$$\phi(t+h, X_{t+h}) - \phi(t, X_t) = \int_t^{t+h} \Big(\frac{\partial \phi}{\partial t} + b\frac{\partial \phi}{\partial x} + \frac{\sigma^2}{2}\frac{\partial \phi}{\partial x^2}\Big)(s, X_s)ds + \int_t^{t+h} \Big(\sigma\frac{\partial \phi}{\partial x}\Big)(s, X_s)dW_s$$

For convenience, set $f_1(t,x) := \Big(\frac{\partial \phi}{\partial t} + b\frac{\partial \phi}{\partial x} + \frac{\sigma^2}{2}\frac{\partial \phi}{\partial x^2}\Big)(t,x)$ and $f_2(t,x) := \Big(\sigma\frac{\partial \phi}{\partial x}\Big)(t,x)$

Thus, we can rewrite $\mathcal{G}$ as:

$$\mathcal{G}\phi(t,x) := \lim_{h\to 0} \frac{1}{h}\mathrm{EVaR}_{\beta(t).h}\Big[\int_t^{t+h} f_1(s, X_s)ds + \int_t^{t+h} f_2(s, X_s)dWs \Big| X_t = x\Big]$$

Sketch of Proof

Now, the goal is to show that for each (t,x), the following inequality holds,

$$\left|\mathcal{G}\phi(t,x) - f_1(t,x) - \sqrt{2\beta(t)}|f_2(t,x)|\right| \leq 0$$

Using subadditivity of EVaR and the triangle inequality:

$$\begin{aligned} 0 \leq \lim_{h\to 0} &\left|\mathrm{EVaR}_{\beta(t).h}\left[\frac{1}{h}\int_t^{t+h} f_1(s,X_s)ds - f_1(t,x)\middle| X_t = x\right]\right| \\ + \lim_{h\to 0} &\left|\mathrm{EVaR}_{\beta(t).h}\left[\frac{1}{h}\int_t^{t+h} f_2(s,X_s)dW_s - \sqrt{2\beta(t)}|f_2(t,x)|\middle| X_t = x\right]\right| \end{aligned} \tag{3}$$

Consider each term separately.

Sketch of Proof

In the case of the first term, note that $s \mapsto f_1(s, X_s) - f_1(t, x)$ is continuous almost surely and hence the mean value theorem for definite integrals implies that there exists a $\xi \in [t, t+h]$ such that

$$\frac{1}{h}\int_t^{t+h} f_1(s, X_s)ds - f_1(t, x) = f_1(\xi, X_\xi) - f_1(t, x)$$

Then, the subadditivity of EVaR yields,

$$\lim_{h\to 0}\left|\mathrm{EVaR}_{\beta(t).h}\Big(\frac{1}{h}\int_t^{t+h} f_1(s, X_s) - f_1(t, x)ds\Big|X_t = x\Big)\right| = 0$$

Sketch of Proof

In the case of the second term, note that we have the following bound,

$$\mathrm{EVaR}_{\beta(t).h}\Big[\frac{1}{h}\int_t^{t+h} f_2(s,X_s)dW_s\Big|X_t=x\Big] \leq$$
$$\mathrm{EVaR}_{\beta(t).h}\Big[\frac{1}{h}\int_t^{t+h} f_2(s,X_s)-f_2(t,x)dW_s\Big|X_t=x\Big]+$$
$$\mathrm{EVaR}_{\beta(t).h}\Big[\frac{1}{h}\int_t^{t+h} f_2(t,x)dW_s\Big|X_t=x\Big]$$

where $\mathrm{EVaR}_{\beta(t).h}\Big[\frac{1}{h}\int_t^{t+h} f_2(t,x)dW_s\Big|X_t=x\Big]=\sqrt{2\beta(t)}|f_2(t,x)|$. Thus,

$$(3) \leq \lim_{h\to 0}\Big|\mathrm{EVaR}_{\beta(t).h}\Big[\frac{1}{h}\int_t^{t+h} f_2(s,X_s)-f_2(t,x)dW_s\Big|X_t=x\Big]\Big|$$

Sketch of Proof

Therefore, it only remains to show that:

$$\lim_{h\to 0}\left|\mathrm{EVaR}_{\beta(t).h}\left[\frac{1}{h}\int_t^{t+h} f_2(s,X_s)-f_2(t,x)dW_s\middle|X_t=x\right]\right|=0$$

Form the stochastic integral $M_h := \int_t^{t+h} f_2(s,X_s)-f_2(t,x)dW_s$, which is a continuous martingale with bounded quadratic variation,

$$\langle M\rangle_h \le \frac{\hat{C}^2 h^{1+2\eta}}{2\eta+1} \tag{4}$$

Then, according to the definition of the EVaR,

$$\frac{1}{h}\mathrm{EVaR}_{\beta(t).h}(M_h|X_t=x)=\inf_{\ell>0}\frac{1}{h\ell}\Big(\beta(t).h+\log(\mathbb{E}[\exp(\ell M_h)])|X_t=x\Big)$$

Sketch of Proof

But M_h is a martingale and satisfies Novikov's condition and thus $\mathbb{E}[\exp(\ell M_h - \ell^2 \frac{\langle M \rangle_h}{2})] = 1$. Combining this with (4) yield,

$$\mathbb{E}[e^{\ell M_h}] \le \exp\left(\frac{\ell^2}{2}.\frac{\hat{C}^2 h^{2\eta+1}}{2\eta+1}\right)$$

Therefore,

$$\frac{1}{h}\mathrm{EVaR}_{\beta(t).h}(M_h | X_t = x) \le \inf_{\ell>0} \frac{1}{h\ell}\left(\beta(t).h + \frac{\ell^2}{2}.\frac{\hat{C}^2 h^{2\eta+1}}{2\eta+1}\right)$$
$$= \inf_{\ell>0} \frac{\beta(t)}{\ell} + \frac{\ell}{2}.\frac{\hat{C}^2 h^{2\eta}}{2\eta+1} = \sqrt{2\beta(t)}.\frac{\hat{C} h^{\eta}}{\sqrt{2\eta+1}}$$

We conclude that:

$$\lim_{h\to 0}\left|\mathrm{EVaR}_{\beta(t).h}\left[\frac{1}{h}\int_t^{t+h} f_2(s, X_s) - f_2(t,x) dW_s \middle| X_t = x\right]\right| = 0$$

which completes the proof.

Formulation of Risk-averse Optimal Control Problem

After derivation of risk-averse generator, it is time to formulate the risk-averse optimal control problem and derive the associated HJB equation.

$\rightarrow$ Recall the classical optimal control setup:

Consider the set of admissible controls

$$\mathcal{U}[0, T] := \{u : [0, T] \times \Omega \rightarrow U | u \text{ is adapted}\}$$

where $U \subset \mathbb{R}$. Consider the controlled stochastic process $(X_s^{t,x,u})$ given by: $(u \in \mathcal{U}[t, T])$

$$\begin{aligned} dX_s^{t,x,u} &= b(s, X_s^{t,x,u}, u(s))ds + \sigma(s, X_s^{t,x,u}, u(s))dW_s \\ X_t^{t,x,u} &= x \end{aligned} \tag{5}$$

For the running cost $h : [0, t] \times \mathbb{R} \times U \rightarrow \mathbb{R}$ and a terminal cost $g : \mathbb{R} \rightarrow \mathbb{R}$, the total cost accumulated over $[t, T]$ is

$$\int_t^T h(s, X_s^{t,x,u}, u(s))ds + g(X_T^{t,x,u})$$

Formulation of Risk-averse Optimal Control Problem

→ Now, the risk-averse optimal control problem formulated as follow:

For $\beta : [0, T] \rightarrow [0, \infty)$ define the controlled value function:

$$V^u(t,x) := \mathrm{nEVaR}^{t:T}_{\beta(.)}\Big(\int_t^T h(s, X_s^{t,x,u}, u(s))ds + g(X_T^{t,x,u}) \Big)$$

Then, optimal value function $V : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$,

$$V(t,x) := \inf_{u \in \mathcal{U}[t,T]} V^u(t,x)$$

→ The next step is to show that the risk-averse optimal value function V satisfies the analogue of the dynamic programming principle.

Dynamic Programming Principle

Lemma 1

Let $(t, x) \in [0, T) \times \mathbb{R}$ and $r \in (t, T]$, then it holds that,

$$V(t,x) = \inf_{u \in \mathcal{U}(t,r)} nEVaR^{t:r}_{\beta(.)}\left(\int_t^r h(s, X_s^{t,x,u}, u_s)ds + V(r, X_r^{t,x,u}) \middle| X_t = x \right) \tag{6}$$

Proof: The proof consists of two steps. In the first step, we start with $V(t,x)$ and show that it is greater than or equal to the RHS. While, the step 2 consists of showing the converse inequality by starting with the RHS.

Step 1:

For every $\varepsilon > 0$ there exist a $\tilde{u}(.) \in \mathcal{U}[t, T]$ such that $V(t,x) + \varepsilon \geq V^{\tilde{u}}(t,x)$. On the other hand, based on the definition of the nested risk measures:

Proof

$$V^{\tilde{u}}(t,x) =$$

$$\text{nEVaR}^{t:r}_{\beta(.)}\Big(\text{nEVaR}^{r:T}_{\beta(.)}\Big(\int_t^T h(s, X_s^{t,x,\tilde{u}}, \tilde{u}_s)ds + g(X_T^{t,x,\tilde{u}})\Big|\mathcal{F}_r\Big)\Big|X_t = x\Big)$$

For each $r \in (t,\, T]$, the following inequality holds almost surely,

$$\text{nEVaR}^{r:T}_{\beta(.)}\Big(\int_r^T h(s, X_s^{r,x(r),\tilde{u}}, \tilde{u}_s)ds + g(X_T^{r,x(r),\tilde{u}})\Big|\mathcal{F}_r\Big) \geq V(r, X_r^{t,x,\tilde{u}})$$

Therefore, subadditivity of nEVaR yields,

$$V(t,x)+\varepsilon \geq \inf_{u\in\mathcal{U}(t,r)} \text{nEVaR}^{t:r}_{\beta(.)}\Big(\int_t^r h(s, X_s^{t,x,u}, u_s)ds + V(r, X_r^{t,x,u})\Big|X_t = x\Big)$$

The fact that $\varepsilon > 0$ can be chosen arbitrarily, completes the step 1 proof $(\geq)$.

Proof

Step 2:

To prove the converse inequality, again consider a fixed $\varepsilon > 0$, then there exist a $\bar{u} \in \mathcal{U}[t,r]$ such that,

$$\inf_{u\in\mathcal{U}(t,r)} \mathrm{nEVaR}^{t:r}_{\beta(.)}\Big(\int_t^r h(s, X_s^{t,x,u}, u_s)ds + V(r, X_r^{t,x,u})\Big| X_t = x\Big) + \varepsilon$$
$$\geq \mathrm{nEVaR}^{t:r}_{\beta(.)}\Big(\int_t^r h(s, X_s^{t,x,\bar{u}}, \bar{u}_s)ds + V(r, X_r^{t,x,\bar{u}})\Big| X_t = x\Big)$$

Then, we need to define the piecewise control function for $[t, r)$ and $[r, T]$. For doing this, for every $y \in \mathbb{R}$ let $\tilde{u}(y) \in \mathcal{U}[r, T]$ be such that $V(r, y) + \varepsilon \geq V^{\tilde{u}(y)}(r, y)$. We assume that the mapping $y \mapsto \tilde{u}(y)$ is measurable and construct the control function:

$$u_s^0 = \begin{cases} \bar{u}_s & s \in [t, r) \\ \tilde{u}_s(X_r^{t,x,\bar{u}}) & s \in [r, T] \end{cases}$$

Proof

Using monotonicity of nested risk measure,

$$
\begin{aligned}
&\mathrm{nEVaR}^{t:r}_{\beta(.)}\Big(\int_t^r h(s, X_s^{t,x,\bar{u}}, \bar{u}_s)ds + V(r, X_r^{t,x,\bar{u}}) \Big| X_t = x \Big) \\
\geq &\mathrm{nEVaR}^{t:r}_{\beta(.)}\Big(\int_t^r h(s, X_s^{t,x,\bar{u}}, \bar{u}_s)ds + V^{\tilde{u}_s(X_r^{t,x,\bar{u}})}(r, X_r^{t,x,\bar{u}}) \Big| X_t = x \Big) - \varepsilon \\
= &\mathrm{nEVaR}^{t:T}_{\beta(.)}\Big(\int_t^T h(s, X_s^{t,x,u^0})ds + g(X_T^{t,x,u^0}) | X_t = x \Big) - \varepsilon \\
&V^{u_0}(t,x) - \varepsilon
\end{aligned}
$$

Combining the inequalities yields,

$$
\begin{aligned}
\inf_{u \in \mathcal{U}(t,r)} \mathrm{nEVaR}^{t:r}_{\beta(.)}\Big(\int_t^r h(s, X_s^{t,x,u}, u_s)ds + V(r, X_r^{t,x,u}) \Big| X_t = x \Big) + \varepsilon \\
\geq V^{u_0}(t,x) - \varepsilon \geq V(t,x) - \varepsilon
\end{aligned}
$$

Again, the fact that ε was arbitrary, finishes the proof.

Hamilton Jacobi Bellman equation

$\rightarrow$ Following theorem provides the only remaining ingredient for deriving the analogue of the HJB equation,

Theorem 2

Let $(X_s)_{s\in\mathcal{T}}$ be the solution of the SDE $dX_t = b(X_t,t)dt + \sigma(X_t,t)dW_t$ with initial condition $X_t = x$. Also, let $h(.,.), V(.,.) \in C^{1,2}(\mathcal{T},\mathbb{R})$ such that $\frac{\partial V}{\partial x}$ is bounded, then it holds (for entropic generator $\mathcal{G}$ derived in Proposition (3)),

$$\lim_{\delta\to 0}\frac{1}{\delta} nEVaR^{t:t+\delta}_{\beta(.)}\left(\int_t^{t+\delta} h(s,X_s)ds + V(t+\delta,X_{t+\delta}) - V(t,x)\Big| X_t = x\right) = h(t,x) + \mathcal{G}V(t,x) \quad (7)$$

Hamilton Jacobi Bellman equation

Taking the limit $r \to t$ in the dynamic programming principle (6) yields,

$$0 = \inf_{u \in \mathcal{U}(t,r)} \frac{1}{r-t} \mathrm{nEVaR}_{\beta(.)}^{t:r}\Big(\int_t^r h(s, X_s^{t,x,u}, u(s))ds + V(r, X_r^{t,x,u}) - V(t,x) \Big| X_t = x \Big)$$

Combining with theorem, results in:

$$\begin{aligned} 0 &= \inf_u \{h(t,x,u) + \mathcal{G}V(t,x)\} \\ &= \inf_u \Big\{ h(t,x,u) + \frac{\partial V}{\partial t}(t,x) + b(t,x,u)\frac{\partial V}{\partial x}(t,x) + \frac{\sigma^2(t,x,u)}{2}\frac{\partial^2 V}{\partial x^2} \\ &\qquad + \sqrt{2\beta(t,x)}\Big|\sigma(t,x,u)\frac{\partial V}{\partial x}(t,x)\Big| \Big\} \end{aligned}$$

Hamilton Jacobi Bellman equation

Therefore,

Corollary 3

the risk-averse value function $V(t,x)$ solves the HJB equation,

$$\frac{\partial V}{\partial t} + \mathcal{H}(t, x, \frac{\partial V}{\partial x}, \frac{\partial^2 V}{\partial x^2}) = 0$$
$$V(T,x) = g(x)$$

with Hamiltonian function

$$\mathcal{H}(t,x,p,w) := \inf_u \left\{ b(u).p + \frac{1}{2}\sigma(u)^2.w + h(u) + \sqrt{2\beta}.|\sigma(u)p| \right\}$$

Proof of Theorem (2)

The idea of proof is to show that:

$$\lim_{\delta\to 0}\left|\frac{1}{\delta}\mathrm{nEVaR}^{t:t+\delta}_{\beta(.)}\left(\int_t^{t+\delta} h(s,X_s)ds + V(t+\delta,X_{t+\delta}) - V(t,x)\Big|\mathcal{F}_t\right) - h(t,x) - \mathcal{G}V(t,x)\right| = 0$$

Also, recall that the entropic generator is defined as,

$$\mathcal{G}V(t,x) = \lim_{\delta\to 0}\frac{1}{\delta}\mathrm{EVaR}_{\beta(t).\delta}\Big(V(t+\delta,X_{t+\delta}|X_t = x) - V(t,x)\Big)$$

Let start with LHS of (7), which can be bounded from above using the subadditivity of coherent risk measures:

$$\lim_{\delta\to 0}\mathrm{nEVaR}^{t:t+\delta}_{\beta(.)}\left(\frac{1}{\delta}\int_t^{t+\delta} h(s,X_s)ds\Big|X_t = x\right) + \lim_{\delta\to 0}\mathrm{nEVaR}^{t:t+\delta}_{\beta(.)}\left(\frac{V(t+\delta,X_{t+\delta}) - V(t,x)}{\delta}\Big|X_t = x\right)$$

Proof of Theorem (2)

where the first part converges to $h(t,x)$ following the arguments in the proof of entropic generator.
The second term can be written as a limit over all possible partitions $\mathcal{P}$ of $[t, t+\delta]$ given by $\mathcal{P} = (t_0, ..., t_n)$

$$\lim_{\delta\to 0}\lim_{\|\mathcal{P}\|\to 0} \mathrm{EVaR}_{\beta(t_0)\Delta t_0}\Bigg(\dots \mathrm{EVaR}_{\beta(t_{n-1})\Delta t_{n-1}}\left(\frac{V(t+\delta, X_{t+\delta}) - V(t,x)}{\delta}\Big|\mathcal{F}_{t_{n-1}}\right)\dots\Big|\mathcal{F}_{t_0}\Bigg)$$

this limit converges uniformly in δ for fixed $\mathcal{P}$ and thus can be interchanged. Then, definition of the entropic generator yields,

$$\lim_{\delta\to 0} \mathrm{nEVaR}^{t:t+\delta}_{\beta(.)}\left(\frac{V(t+\delta, X_{t+\delta}) - V(t,x)}{\delta}\Big|\mathcal{F}_{t_0}\right) = \lim_{\delta\to 0} \mathrm{EVaR}_{\beta(t_0)\delta}\left(\frac{V(t+\delta, X_{t+\delta}) - V(t,x)}{\delta}\Big|\mathcal{F}_{t_0}\right) = \mathcal{G}V(t,x)$$

which leads to the desired expression and terminates the proof.

Table of Contents

Classical Merton's Portfolio Problem

- In the formulation of the classical Merton's portfolio problem, one need to determine the running cost h and the terminal cost g.
- This can be done by applying the concept of utility functions, which characterize the investors preferences and is the function of wealth or consumption.
- Utility function express how satisfied the investor is with a certain outcome of the investment.
- In this setup, running cost h is the utility of consumption, which represents the case where the investor makes a living from the investment and consumes money from the bank account.
- If there is no consumption, then we only need to maximize the expected utility of the final wealth g.

Classical Merton's Portfolio Problem

- It is assumed that utility functions are strictly increasing, strictly concave and continuous. (Tikosi [6])
- Examples: $g(x) = \log(x)$ and $g(x) = x^{\gamma}$ where $0 < \gamma < 1$.
- Consider the case where $g(x) = log(x)$ and there is no consumption by the investor.

→ Recall the controlled SDE:

$$dX_s = (a_1 u_s + r(1 - u_s))X_s ds + a_2 u_s X_s dW_s, \quad \text{for } s > t$$
$$X_t = x$$

Then, our goal is to solve the following equation and obtain the optimal control function:

$$V(t,x) = \max_u \ \mathbb{E}[\log(X(T))|X_t = x]$$

Classical Merton's Portfolio Problem

Applying the Itô formula, integration, and taking expected value yields,

$$\max_u \mathbb{E}\Big[\log(X(T))\Big|X_t = x\Big] = \log(x) + \max_u \mathbb{E}\Big[\int_t^T a_1 u_s + r(1-u_s) - \frac{1}{2}a_2^2 u_s^2 ds\Big]$$

However, the function $a_1 u_s + r(1-u_s) - \frac{1}{2}a_2^2 u_s^2$ is strictly concave:

$$\frac{\partial}{\partial u}\Big(a_1 u_s + r(1-u_s) - \frac{1}{2}a_2^2 u_s^2\Big) = a_1 - r - a_2^2 u$$
$$\frac{\partial^2}{\partial u^2}\Big(a_1 u_s + r(1-u_s) - \frac{1}{2}a_2^2 u_s^2\Big) = -a_2^2 < 0$$

Therefore, we obtain $\hat{u} = \frac{a_1 - r}{a_2^2}$. Thus, the optimal control is defined as,

$$u^* = \begin{cases} 0 & \text{if} \quad \hat{u} < 0 \\ \hat{u} & \text{if} \quad \hat{u} \in [0,1] \\ 1 & \text{if} \quad \hat{u} > 1 \end{cases}$$

Classical Merton's Portfolio Problem

Finally, the corresponding value function is:

$$V(t,x) = \begin{cases} \log(x) + r(T-t) & \text{if} \quad \hat{u} < 0 \\ \log(x) + \left(r + \frac{(a_1 - r)^2}{2a_2^2}\right)(T-t) & \text{if} \quad \hat{u} \in [0,1] \\ \log(x) + \left(a_1 - \frac{1}{2}a_2^2\right)(T-t) & \text{if} \quad \hat{u} > 1 \end{cases}$$

Notice that the simplicity of this solution is due to the fact that the SDE has an exponential solution. Thus, taking the logarithm leads us to the simple equation.

$\rightarrow$ The next step is to compute the associated formula for the $g(x) = log(x)$ in the risk-averse formulation.

Risk-Averse Merton's Portfolio Problem

Derivation by applying the HJB equation: Recall the drift and diffusion term in the controlled SDE were given by:

$$b(s,X) = \big(a_1 u_s + r(1-u_s)\big)X_s, \qquad \sigma(s,X) = a_2 u_s X_s$$

Also, recall the risk-averse Hamiltonian:

$$\mathcal{H}(t,x,V_x,V_{xx}) = \sup_{u\in[0,1]} \left\{ bV_x + \frac{1}{2}\sigma^2 V_{xx} - \sqrt{2\beta_s}|\sigma V_x| \right\}$$

Plug the values for drift and diffusion terms in Hamiltonian:

$$\mathcal{H}(t,x,V_x,V_{xx}) = \sup_{u\in[0,1]} \left\{ \big(a_1 u_s + r(1-u_s)\big)xV_x + \frac{1}{2}a_2^2 u_s^2 x^2 V_{xx} - \sqrt{2\beta_s}|a_2 u_s x V_x| \right\}$$

Under the assumption that $V_x \geq 0,$ and $V_{xx} \leq 0$, the function inside the suprimum is concave, and its first derivative with respect to the control u is:

Risk-Averse Merton's Portfolio Problem

$$a_1 x V_x - r x V_x + a_2^2 u_s x^2 V_{xx} - \sqrt{2\beta_s} a_2 x V_x$$

Thus, the optimal solution is given by:

$$\hat{u}_s = \frac{r - a_1 - a_2\sqrt{2\beta_s}}{a_2^2} \frac{V_x^2}{x V_{xx}} \tag{8}$$

Then, the optimal control is given by,

$$u_t^* = \begin{cases} 0 & \text{if} \quad \hat{u}_t < 0 \\ \hat{u}_t & \text{if} \quad \hat{u}_t \in [0,1] \\ 1 & \text{if} \quad \hat{u}_t > 1 \end{cases}$$

The optimal yields the HJB equation:

$$V_t + \mathcal{H}(t, x, V_x, V_{xx}) = 0$$
$$V(T, x) = log(x)$$

where,

Risk-Averse Merton's Portfolio Problem

$$\mathcal{H}(t,x,V_x,V_{xx}) = \begin{cases} rxV_x & \text{if} \quad \hat{u}_t < 0 \\ rxV_x - \frac{1}{2}\frac{\left(a_1 - r - a_2\sqrt{2\beta_t}\right)^2}{a_2^2}\frac{V_x^2}{V_{xx}} & \text{if} \quad \hat{u}_t \in [0,1] \\ a_1 x V_x + \frac{a_2^2 x^2 V_{xx}}{2} & \text{if} \quad \hat{u}_t > 1 \end{cases}$$

The next step is to find the value function. For the case where $\hat{u}_t \in [0,1]$, we apply the following ansatz:

Ansatz: $V(t,x) = log(x) + f(t)$

Therefore,

$$V_t = f'(t), \qquad V_x = \frac{1}{x}, \qquad V_{xx} = -\frac{1}{x^2}$$

By plugging these values in the HJB equation:

$$f'(t) = \frac{\left(a_1 - r - a_2\sqrt{2\beta_t}\right)^2}{2a_2^2} + r$$

Risk-Averse Merton's Portfolio Problem

Then, the solution to this ODE is given by,

$$f(t) = \int_t^T r + \frac{\left(a_1 - r - a_2\sqrt{2\beta_s}\right)^2}{2a_2^2} ds$$

Consequently,

$$V(t,x) = \log(x) + \int_t^T r + \frac{\left(a_1 - r - a_2\sqrt{2\beta_s}\right)^2}{2a_2^2} ds$$

Similarly, one can apply the same ansatz and obtain the corresponding value function for other two cases:

$$V(t,x) = \begin{cases} \log(x) + r(T-t) & \text{if} \quad \hat{u}_t < 0 \\ \log(x) + \int_t^T r + \frac{\left(a_1 - r - a_2\sqrt{2\beta_s}\right)^2}{2a_2^2} ds & \text{if} \quad \hat{u}_t \in [0,1] \\ \log(x) + \left(a_1 - \frac{a_2^2}{2}\right)(T-t) & \text{if} \quad \hat{u}_t > 1 \end{cases}$$

Risk-Averse Merton's Portfolio Problem

Finally, plugging $V(t,x)$ into the equation (8), leads to the following control $\hat{u}_t = \frac{a_1 - r - a_2\sqrt{2\beta_t}}{a_2^2}$ for $\hat{u} \in [0,1]$. Then, the optimal control is given by,

$$u_t^* = \begin{cases} 0 & \text{if} \quad \hat{u}_t < 0 \\ \hat{u}_t & \text{if} \quad \hat{u}_t \in [0,1] \\ 1 & \text{if} \quad \hat{u}_t > 1 \end{cases}$$

which matches the result of the first derivation.

General Risk-Neutral Merton's Portfolio Problem

→ The next step is to consider the more general case where the running cost h is not zero. In this case, we assume that the investor consumes wealth at non-negative rate $c(t)$ at time $t \geq 0$.
Therefore the controlled SDE becomes,

$$dX_s = [(a_1 u_s + r(1-u_s))X_s - c_s]ds + a_2 u_s X_s dW_s, \quad \text{for } s > t$$
$$X_t = x$$

Also, assume that the running utility of consumption is given by $h(c) = \frac{c^p}{p}$ and the utility derived from terminal wealth is of the form $g(x) = \frac{x^p}{p}$, where $0 < p < 1$. Then, the Hamiltonian is,

$$\mathcal{H}(t, x, V_x, V_{xx}) =$$
$$\sup_{u \in [0,1], c} \left\{ \left[\left(a_1 u_s + r(1-u_s)\right)x - c_s\right]V_x + \frac{1}{2}a_2^2 u_s^2 x^2 V_{xx} + \frac{c_s^p}{p} \right\}$$

General Risk-Neutral Merton's Portfolio Problem

Therefore,

$$\mathcal{H}(t, x, V_x, V_{xx}) =$$

$$\sup_{u\in[0,1]} \left\{ \big(a_1 u_s + r(1-u_s)\big) x V_x + \frac{1}{2} a_2^2 u_s^2 x^2 V_{xx} \right\} + \sup_c \left\{ \frac{c_s^p}{p} - c_s V_x \right\}$$

Thus, the optimal controls are given by,

$$\hat{u}_s = \frac{r - a_1}{a_2^2} \frac{V_x^2}{x V_{xx}} \ , \quad c_s^* = V_x^{\frac{1}{p-1}}$$

Define,

$$u_t^* = \begin{cases} 0 & \text{if} \quad \hat{u}_t < 0 \\ \hat{u}_t & \text{if} \quad \hat{u}_t \in [0,1] \\ 1 & \text{if} \quad \hat{u}_t > 1 \end{cases}$$

Plug these values in HJB equation (for $\hat{u} \in [0, 1]$ case)

General Risk-Neutral Merton's Portfolio Problem

$$V_t + rxV_x - \frac{1}{2}\frac{(a_1 - r)^2}{a_2^2}\frac{V_x^2}{V_{xx}} + \frac{1-p}{p}V_x^{\frac{p}{p-1}} = 0$$
$$V(T, x) = \frac{x^p}{p}$$

The objective is to find the value function that satisfies the HJB equation. For doing this, use the following ansatz,

$$V(t, x) = f(t)^{1-p}\frac{x^p}{p}$$

Then,

$$V_t = (1-p)f(t)^{-p}f'(t)\frac{x^p}{p}, \quad V_x = f(t)^{1-p}x^{p-1}, \quad V_{xx} = (p-1)f(t)^{1-p}x^{p-2}$$

General Risk-Neutral Merton's Portfolio Problem

By plugging into HJB equation,

$$\left(\frac{1-p}{p}f(t)^{-p}f'(t) + rf(t)^{1-p} - \frac{1}{2(p-1)}\frac{(a_1-r)^2}{a_2^2}f(t)^{1-p} + \frac{1-p}{p}f(t)^{-p}\right)x^p = 0$$

$$f(T)^{1-p} = 1$$

Mathematical simplification yields the following ODE,

$$f'(t) + \frac{1}{1-p}\left(rp - \frac{p}{2(p-1)}\frac{(a_1-r)^2}{a_2^2}\right)f(t) + 1 = 0$$

$$f(T) = 1$$

Let $k = \frac{1}{1-p}\Big(-rp - \frac{p}{2(1-p)}\frac{(a_1-r)^2}{a_2^2}\Big)$. Then, the ODE becomes,

General Risk-Neutral Merton's Portfolio Problem

$$f'(t) - kf(t) + 1 = 0$$
$$f(T) = 1$$

And the solution to this ODE is given by,

$$f(t) = \frac{1 - \exp\{-k(T-t)\}}{k} + \exp\{-k(T-t)\}$$

The final step is to compute the optimal controls using this choice of value function V,

$$\hat{u} = \frac{a_2 - r}{a_2^2(1-p)}, \qquad c^*(t) = \frac{X(t)}{f(t)}$$

$\rightarrow$ Therefore, the investor should trade continuously in order to keep a constant fraction u^* of wealth in the stocks and consume at the rate proportional to total wealth; however, the proportion is time-dependent.

General Risk-Averse Merton's Portfolio Problem

$\rightarrow$ Now, the objective is to extend the general Merton's portfolio problem to the risk-averse case.
Recall that,

$$b(s,X) = \big(a_1 u_s + r(1-u_s)\big)X_s - c_s \qquad \sigma(s,X) = a_2 u_s X_s$$

Then, the Hamiltonian is given by,

$$\mathcal{H}(t,x,V_x,V_{xx}) =$$

$$\sup_{u\in[0,1],c} \left\{ \big(a_1 u_s + r(1-u_s)\big)xV_x - c_s V_x + \frac{1}{2}a_2^2 u_s^2 x^2 V_{xx} - \sqrt{2\beta_s}|a_2 u_s x V_x| + \frac{c_s^p}{p} \right\}$$

Thus,

General Risk-Averse Merton's Portfolio Problem

$$\mathcal{H}(t,x,V_x,V_{xx}) = \sup_{u\in[0,1]}\left\{\big(a_1u_s + r(1-u_s)\big)xV_x + \frac{1}{2}a_2^2u_s^2x^2V_{xx} - \sqrt{2\beta_s}|a_2u_sxV_x|\right\} + \sup_{c}\left\{+\frac{c_s^p}{p} - c_sV_x\right\}$$

Then, again under the assumption that $V_X \geq 0$ and $V_{xx} \leq 0$ inside the first suprimum is concave. Then, the optimal solutions are given by,

$$\hat{u}_s = \frac{r - a_1 - a_2\sqrt{2\beta_s}}{a_2^2}\frac{V_x^2}{V_{xx}}\,, \qquad c_s^* = V_x^{\frac{1}{p-1}}$$

Plugging these values in HJB equation yields,

General Risk-Averse Merton's Portfolio Problem

$$V_t + rxV_x - \frac{1}{2}\frac{(a_1 - r - a_2\sqrt{2\beta_t})^2}{a_2^2}\frac{V_x^2}{V_{xx}} + \frac{1-p}{p}V_x^{\frac{p}{p-1}} = 0$$
$$V(T,x) = \frac{x^p}{p}$$

The next step is to obtain the solution of the HJB,

Ansatz: $V(t,x) = f(t)^{1-p}\frac{x^p}{p}$

Plug into HJB and some mathematical simplification yields the following ODE:

$$f'(t) - k(t)f(t) + 1 = 0$$
$$f(T) = 1$$

where $k(t) = \frac{1}{1-p}\Big(- rp - \frac{p}{2(1-p)}\frac{(a_1 - r - a_2\sqrt{2\beta_t})^2}{a_2^2}\Big)$.

General Risk-Averse Merton's Portfolio Problem

Then, the solution to this ODE is given by,

$$f(t) = \frac{1 - \int_t^T \exp\{-\int_t^T k(s)ds\}ds}{\exp\{-\int_t^T k(s)ds\}}$$

Remark

This derivation is for the general case, where the risk level β is function of time. If β is constant, then the value function is similar to the risk-neutral case, of course with new k.

The final step is to compute the optimal controls using this choice of value function V,

$$\hat{u}(t) = \frac{a_1 - r - a_2\sqrt{2\beta_t}}{a_2^2(1-p)}, \qquad c^*(t) = \frac{X(t)}{f(t)}$$

References

1. Artzner, Philippe and Delbaen, Freddy and Eber, Jean-Marc and Heath, David (1999)
Coherent measures of risk

2. Pichler, Alois and Schlotter, Ruben (2019)
Risk-Averse Optimal Control

3. Pichler, Alois and Schlotter, Ruben (2019)
Risk-Averse Dynamic Optimization

4. Pichler, Alois and Schlotter, Ruben and Shapiro, Alexander (2019)
Uncertainty quantification: dynamic optimization with risk

5. Ahmadi-Javid, Amir (2012)
Entropic value-at-risk: A new coherent risk measure

6. Kinga Tikosi(2016)
Merton's Portfolio Problem